COMMUTATIVE ALGEBRA M-305.2

UNIT-I

Ring and Ring Homomorphism, Ideals, Quotient rings, Zero divisons, Nilpotent elements, Units, Prime Ideals and Maxim-l Ideals, Nil radical, Jacobson radical, Operations on idea Extensions and Contractions.

UNIT-II

Modules and Module homomorphism, Submodules and Quotien modules, Operations on Submodules, Direct Sum and product, Finitely Generated modules, Exact Sequencies, Tensor product of modules, Restriction and Extension of scalars, Exactness Properties of the tensor product, Algebras, Tensor product of algebras

UNIT-III

Local properties, Extended and Contracted ideals in rings of fractions

UNIT-IV

Primary Decompositions

Textbook:- Introduction to Commutative Algebra

- M. F. Atiyah

Unit - I

Ring:- Let 'R' be a non-empty set and $+, \cdot$ be binary operations defined on 'R' then an algebraic structure $(R,+,\cdot)$ is called a ring if

i, $(R,+)$ is an abelian group

ii, $(R,\cdot)$ is a semi group

iii, Distributive laws hold

$$a\cdot(b+c) = ab+ac$$

$$(a+b)\cdot c = ac+bc \quad \forall\, a,b,c \in R$$

Ring with identity:- Let $(R,+,\cdot)$ be the ring then there exists $1 \in R$ such that $1\cdot a = a = a\cdot 1 \;\forall\, a \in R$ then '1' is called identity element of 'R' and $(R,+,\cdot)$ is called ring with identity element.

Ring Homomorphism:- Let R, R' be rings then the mapping $f: R \longrightarrow R'$ is called a ring homomorphism if

i, $f(x+y) = f(x)+f(y)$

ii, $f(xy) = f(x)\cdot f(y)$

iii, $f(1) = 1 \;\forall\, 1, x, y \in R$ (or)

$$f(ax+by) = a\cdot f(x) + b f(y)$$

Subring:- Let $(R,+,\cdot)$ be the ring and 'S' be the non-empty subset of the ring 'R' then 'S' is called subring if 'S' itself as a ring.

Ideal:- Let $(R,+,\cdot)$ be the ring and 'I' be the non-empty subset of the ring 'R' is called an ideal if

i, $a, b \in I \Rightarrow a-b \in I$

ii, $a \in I,\; r \in R \Rightarrow ar, ra \in I$

Note:- An ideal is always a subring but the converse is not true.

Prime Ideal:- A proper ideal 'p' of a ring 'R' is called a p-
-e ideal if $xy \in P \Rightarrow$ either $x \in P$ (or) $y \in P$ $\forall x, y \in R$

(or)

A proper ideal 'p' of a ring 'R' is called Prime ideal if $AB \subseteq P \Rightarrow$ either $A \subseteq P$ (or) $B \subseteq P$ for any two ideals A, B of a ring R.

Quotient Ring:- Let $(R, +, \cdot)$ be the ring and 'I' be an ideal of a ring 'R' then $\frac{R}{I} = \{x + I / x \in R\}$ is called a "Quotient Ring" if $\frac{R}{I}$ itself as a ring.

The binary operations '+' and '·' defined on $\frac{R}{I}$ is given by

$(x+I) + (y+I) = (x+y) + I$

$(x+I) \cdot (y+I) = xy + I \quad \forall x, y \in R$

Ring with zero divisors:- A non-zero element $a \in R$ is called a zero divison of 'R' if $\exists$ a non-zero element $b \in R \ni$ $a \cdot b = 0$ (e.g: matrices with zeros)

Ring without zero divisors:- Two elements 'a' and 'b' in a ring 'R' is called without zero divisors if $a \cdot b = 0 \Rightarrow a = 0$ or $b = 0$

Field:- A ring 'R' with atleast two elements is called a field if

i) it is commutative

ii) It has unity

iii) Every non-zero element possess multiplicative inverse.

Skew field or Divison ring:- A ring 'R' with atleast two elements is called skew field if

i) It has unity.

ii) Every non-zero element possess multiplicative inverse.

Note:- i) A commutative divison ring is a field.

ii) Every field is an Integral Domain

Integral domain:- A ring 'R' with atleast two elements is called integral domain if

i) it is commutative

ii) It has without zero divi...

Maximal Ideal:- An ideal 'M' of a ring 'R' is called Maximal ideal if (i), $M \neq R$ (ii), $\exists$ an ideal N of $R \ni M \subseteq N \subseteq R \Rightarrow$ either $M = N$

For any two ideals A/B of a ring 'R' (or) $N = R$

Principal Ideal:- The multiples of an element $x \in R$ form a principal ideal ie, $\langle a \rangle, \langle b \rangle, \langle c \rangle, \dots$

$aR, bR, cR \dots$

$ax, bx, cx \dots \forall x \in R$

Unit:- A non-zero element $x \in R$ is called "unit" if $\exists$ a non-zero element $y \in R \ni xy = 1 = yx$

Nilpotent element:- A non-zero element $a \in R$ is called nilpotent element if $a^n = 0$ for some +ve integer 'n'

Principal Ideal Ring:- The ring contains the ideals, all those ideals are all principal ideals.

Result:-1 An ideal 'M' of a commutative ring 'R' is called Maximal iff $\frac{R}{M}$ is a field.

Let us take 'R' be a commutative ring and 'M' is maximal ideal in 'R'

By definition, M is a proper ideal of a ring 'R' ie, $M \neq R$

$\Rightarrow$ ie, $\frac{R}{M} \neq 0$

$\Rightarrow$ ie, $\frac{R}{M}$ is a non-zero ring

Since 'R' is a commutative ring, we have $\frac{R}{M}$ is also commutative ring.

$x + M \in \frac{R}{M}$ where

Now, let us take $x + M \neq 0 + M$

ie, $x + M \neq M$

ie, $x \notin M$

$\therefore$ we have M is maximal and $x \notin M$

Recall:- M is maximal ideal in a ring 'R' $\Leftrightarrow \forall x \notin M$ $\exists$ $y \in R \ni 1 - xy \in M$

By the above recall $\exists$ $y \in R \ni 1 - xy \in M$

$\Rightarrow 1 + M = (xy) + M$

$\Rightarrow 1 + M = (x + M)(y + M)$

$\therefore \exists$ $y + M \in \frac{R}{M} \ni x + M(y + M) = 1 + M$

$\therefore$ Every non-zero element in $\frac{R}{M}$ has a multiplicative inverse.

$\therefore \frac{R}{M}$ is a field.

Conversely, suppose that $\frac{R}{M}$ is a field.

Claim:- M is maximal.

Let us take $x \notin M$

$\Rightarrow x + M \neq M$

$\Rightarrow x + M \neq 0 + M \in \frac{R}{M}$

$\Rightarrow x + M \in \frac{R}{M}$ where $x + M \neq 0 + M$

Since $\frac{R}{M}$ is a field.

By definition of a field every non-zero element possess multiplicative inverse.

$\therefore \exists$ an element $y + M \in \frac{R}{M} \ni (x+M)(y+M) = 1+M$

$(xy + M) = 1 + M$

$\Rightarrow 1 - xy \in M$

$\therefore$ we have $x \notin M \ \exists \ y \in R \ni 1 - xy \in M$

By the above recall, we have M is maximal ideal in a ring 'R'

Result:-2 An ideal 'p' of a ring 'R' is prime $\Leftrightarrow \frac{R}{P}$ is an integral domain

Take 'R' be a commutative ring and 'p' be a prime ideal in a ring 'R' ie, $P \neq R$

ie, $\frac{R}{P} \neq 0$

ie, $\frac{R}{P}$ is a commutative ring.

Let us take $x+P, y+P \in \frac{R}{P} \ni (x+P)(y+P) = 0+P$

$\Rightarrow xy + P = 0 + P = P$

$\Rightarrow xy \in P$

Since 'p' is a prime ideal, by definition either $x \in P$ or $y \in P$

ie, $x + P \in \frac{R}{P}$ or $y + P \in \frac{R}{P}$

ie, $x + P = 0 + P$ or $y + P = 0 + P$

ie, $x + P = P$ or $y + P = P$

$\therefore \frac{R}{P}$ is an integral domain.

Conversely, suppose that $\frac{R}{P}$ is an integral domain

Claim:- ~~$\frac{R}{P}$ is an~~ 'p' is a Prime ideal.

Let us take $xy \in P$ where $x, y \in R$

$\Rightarrow xy + P = P$

$\Rightarrow xy + P = 0 + P$

$\Rightarrow (x+P)(y+P) = 0+P$

Since $\frac{R}{P}$ is an integral domain

By definition, either $x+P = 0+P$ or $y+P = 0+P$

$\Rightarrow x+P = P$ or $y+P = P$

$\Rightarrow$ either $x \in P$ or $y \in P$

$\therefore P$ is a prime ideal.

Hence proved.

Proposition:- 1.1 There is a one to one order preserving correspondence between the set of all ideals of a ring 'A' containing 'I' and the set of all ideals of the ring $\frac{A}{I}$

Proof:- Let us take the set $S = \{J \mid J$ is an ideal of a ring A containing I $\}$ and 'P' be the set of all ideals of the ring $\frac{A}{I}$

Now define a mapping $\phi : S \to P$ given by $\phi(J) = \frac{J}{I}$ where $\frac{J}{I} = \{x + I \mid x \in J\}$

Now we have to show that $\frac{J}{I}$ is an ideal of the ring $\frac{A}{I}$

Let us take $x+I, y+I \in \frac{J}{I}$ where $x, y \in J$, $x-y \in J$ ($\because J$ is ideal)

Now $(x+I) - (y+I) = (x-y)+I \in \frac{J}{I}$

Now $x+I \in \frac{J}{I}$ and $a+I \in \frac{A}{I}$ where $x \in J$, $a \in A$

$xa \in J$ ($\because J$ is an ideal)

consider $(x+I)\cdot(a+I) = xa+I \in \frac{J}{I}$

$\therefore \frac{J}{I}$ is an ideal of the ring $\frac{A}{I}$

$\therefore$ The mapping ϕ is well-defined.

Now prove that ϕ is one-one:-

Let us take $J_1, J_2 \in S \ni \phi(J_1) = \phi(J_2)$

$\Rightarrow \frac{J_1}{I} = \frac{J_2}{I}$

Take $x \in J_1 \Leftrightarrow x+I \in \frac{J_1}{I} = \frac{J_2}{I}$

$\Leftrightarrow x+I \in \frac{J_2}{I}$

$\Leftrightarrow x \in J_2$

$J_1 = J_2$

$\therefore \phi$ is one-one.

Now prove that ϕ is onto:-

Let us take 'K' be an ideal of the ring $\frac{A}{I}$

Also take the set $J = \{x \in A \mid x+I \in K\}$

Now let us take $x, y \in J \Rightarrow x+I, y+I \in K$

Now take 'K' be an ideal of the ring $\frac{A}{I}$.

Prove that 'J' is an ideal of the ring 'A'

Let us take $x, y \in J \Rightarrow x+I, y+I \in K$

$$\Rightarrow (x+I)-(y+I) \in K$$

$$\Rightarrow (x-y)+I \in K$$

$$\Rightarrow x-y \in J$$

Now let $x \in J$ and $a \in A$

$\Rightarrow x+I \in K$ and $a+I \in \frac{A}{I}$

$\Rightarrow (x+I)\cdot(a+I) \in K$ ($\because$ K is an ideal)

$\Rightarrow xa+I \in K$

$\Rightarrow xa \in J$

$\therefore$ J is an ideal of the ring 'A'

Now take an element $x \in I \Rightarrow x+I = I$

$$\Rightarrow x+I = I = 0+I \in K$$

$$\Rightarrow x+I \in K$$

$$\Rightarrow x \in J$$

$\therefore I \subseteq J$

$\therefore$ J is an ideal of the ring 'A' containing 'I'.

Also, we have $\frac{J}{I} = \{x+I \mid x \in J\} = K$

$\therefore \forall K \in P \ \exists \ J \in S \ni \phi(J) = K = \frac{J}{I}$

$$\Rightarrow \phi(J) = \frac{J}{I}$$

$\therefore \phi$ is onto.

There is a one to one correspondence between the set of ideals of a ring 'A' containing 'I' and the set of all ideals of the ring $\frac{A}{I}$.

Hence proved.

Theorem:- 1.2

Let $A \neq 0$ be the ring then the following conditions are equivalent.

i, 'A' is a field.

ii, The only ideals in the ring 'A' are zero ideal (0) (or) Principal ideal one (1)

iii, Every homomorphism of a ring 'A' into the non-zero ring 'B' is injective.

Proof:- Given that $A \neq 0$ be the ring

i $\Rightarrow$ ii:- Let us take 'A' be the field.

Let us take 'I' be an ideal of the ring 'A' and also take $I \neq$

Claim:- $I = \langle 1 \rangle$

Let us take an element $0 \neq a \in I \subseteq A$

ie, $a \in A$ and A is a field.

$\Rightarrow a^{-1} \in A$

Now $a \in I$, $a^{-1} \in A$ and I is an ideal in a ring 'A'

$\Rightarrow a \cdot a^{-1} \in I$

$\Rightarrow 1 \in I$ and I is an ideal in a ring 'A'

*Recall:- $1 \in U$, U is an ideal in a ring 'R' then $U = R$

By the above recall, we have $I = A$

$= 1 \cdot A$

$= \langle 1 \rangle$

$\therefore I = \langle 1 \rangle$

ii $\Rightarrow$ iii :- Let the only ideals in a ring 'A' are zero ideal (0) or Principal ideal $\langle 1 \rangle$

Let us take 'B' be the non-zero ring and $f: A \to B$ is a homomorphism

Claim:- f is one-one.

By definition of homomorphism, we have $f(1) = 1$

We know that $f(1) \in \text{Img} f \neq \ker f$

$\Rightarrow f(1) \notin \ker f$

$\Rightarrow 1 \notin \ker f$

$\Rightarrow \langle 1 \rangle \notin \ker f$

$\Rightarrow \ker f \neq \langle 1 \rangle$

We know that $\ker f$ is always an ideal of a ring (A)

$\therefore$ By the condition ii we must have $\ker f = \langle 0 \rangle$

Recall:- Let $f: A \to B$ be a homomorphism then $\ker f = 0 \Leftrightarrow f$ is o

By the recall, f is one-one.

(iii) $\Rightarrow$ (i):- Take every homomorphism of a ring 'A' into a non

ring 'B' is injective.

Claim:- 'A' is a field.

Let us take $0 \neq a \in A$ and $I = \langle a \rangle$ is an ideal of a ring 'A'

Claim:- $I = A$

Suppose on the contrary, $I \neq A$

$$\Rightarrow \frac{A}{I} \neq 0$$

ie, $\frac{A}{I}$ is a non-zero ring.

Now define a mapping $\phi: A \to \frac{A}{I}$ be a homomorphism $\ni \ker\phi = I$

By the (iii) condition, we have ϕ is one-one.

$\Rightarrow \ker\phi = 0$

$\Rightarrow I = 0$, which is a contradiction.

$\therefore$ our assumption is wrong.

we must have $I = A \Rightarrow 1 \in I = \langle a \rangle$

$\Rightarrow 1 \in \langle a \rangle = A$

$\Rightarrow 1 = ab$ for some $b \in A$

($\phi: A \to \frac{A}{I}$ $\ker\phi \in A$ we take I is an ideal in ring 'A' $\Rightarrow \ker\phi = I$)

$\therefore$ we have $0 \neq a \in A$, $\exists\, b \in A \ni ab = 1$

ie, Every non-zero element possess multiplicative inverse in A

$\therefore A$ is a field.

Hence proved.

Note:- 1, Let $f: G \to G'$ be a group homomorphism then $\ker f$ and

are normal subgroups of G and G' respectively.

2, Let $f: R \to R'$ be a ring homomorphism $\ker f$ and $\operatorname{img} f$ ar

-als of R and R' respectively.

3, $1 \in U$ and U is an ideal of a ring 'R' then $U = R$

4, $\phi: A \to B$ be a homomorphism then $\ker\phi = 0 \Leftrightarrow \phi$ is one-on

Zorn's Lemma:- Every chain in a poset has an upper bound

the poset has a maximal element.

Theorem:- 1.3 Every ring $A \neq 0$ has atleast one maximal ideal.

Proof:- Given that $A \neq 0$ be the ring

Define a set $F = \{I \mid I$ is an ideal in a ring 'A' and $I \neq A\}$

clearly F is a non-empty set

Define a relation '$\subseteq$' on F

Clearly $(F, \subseteq)$ is a poset.

Let us take $\{I_\alpha \mid \alpha \in \Delta\}$ be a chain in a poset $(F, \subseteq)$

Let us take $J = \bigcup_{\alpha \in \Delta} I_\alpha$

Now we have to show that 'J' is an ideal in a ring 'A'

Let us take $x, y \in J = \bigcup_{\alpha \in \Delta} I_\alpha$

$\Rightarrow x \in I_\alpha$ and $y \in I_\beta$ for some $\beta \in \Delta$

without loss of generality, we have either $I_\alpha \subseteq I_\beta$ or $I_\beta \subseteq I_\alpha$

Let us take $I_\alpha \subseteq I_\beta$

We have $x \in I_\alpha \subseteq I_\beta \Rightarrow x \in I_\beta$

Now $x \in I_\beta$, $y \in I_\beta$ and I_β is an ideal in a ring 'A'

$\Rightarrow x - y \in I_\beta$

$\Rightarrow x - y \in \bigcup I_\beta$ for some $\beta \in \Delta$

$\Rightarrow x - y \in \bigcup I_\alpha$ for some $\alpha \in \Delta$

$\Rightarrow x - y \in J$

Now $x \in J$ and $a \in A$

$\Rightarrow x \in \bigcup_{\alpha \in \Delta} I_\alpha$

$\Rightarrow x \in I_\alpha$, $\alpha \in \Delta$

we have $x \in I_\alpha$, $a \in A$ and I_α is an ideal in a ring 'A'

$\Rightarrow xa \in I_\alpha$

$\Rightarrow xa \in \bigcup_{\alpha \in \Delta} I_\alpha$

$\Rightarrow xa \in J$

$\Rightarrow J$ is an ideal in a ring 'A'

Also we have each $I_\alpha \neq A$

$\Rightarrow \bigcup_{\alpha \in \Delta} I_\alpha \neq A$

$\Rightarrow J \neq A$

$\therefore J$ is an ideal in a ring 'A' $\ni J \neq A$

$\therefore J$ is an upper bound for the chain in a poset.

By zorn's lemma the poset has a maximal element say M.
Now we have to show that M is a maximal ideal in a ring 'A'.
Suppose on the contrary, M is not a maximal ideal.
$\therefore \exists$ an ideal 'J' of a ring 'A' $\ni M \subseteq J \subseteq A \Rightarrow M \neq J$ and $J \neq A$
Now we have J is an ideal in a ring 'A' and $J \neq A \Rightarrow J \in F$
Now we have $M, J \in F$ and M is maximal element in $F \Rightarrow J \subseteq M$
But we have $M \subseteq J$
$\Rightarrow M = J$
which is a contradiction to $M \neq J$
This contradiction arises due to our assumption.
$\therefore$ Our assumption is wrong.
$\therefore$ M is maximal ideal in a ring 'A'.

Hence proved.

Corollary:- 1.4 If $I \neq \langle 1 \rangle$ be an ideal of the ring 'A' then there exists a maximal ideal 'M' of a ring 'AR' containing 'I' (or)
If 'I' is a proper ideal of a ring 'R' then $\exists$ a maximal ideal 'M' of a ring containing 'I'
Given that 'R' be the ring and $I \neq \langle 1 \rangle$ be an ideal of a ring 'R'
i.e, $I \neq R$ be a proper ideal of a ring 'R'
Define a set $F = \{J \mid J$ is an ideal of a ring 'R' $\mid J \supseteq I\}$
clearly, F is a non-empty set.
Define a relation $\subseteq$ on a set F
Now we have to show that $(F, \subseteq)$ is a poset.
Reflexive:- Let us take $J_1 \in F$
clearly $J_1 \subseteq J_1$
Anti-Symmetric:- Let $J_1, J_2 \in F$
Now $J_1 \subseteq J_2$ and $J_2 \subseteq J_1 \Rightarrow J_1 = J_2$
Transitive:- Let $J_1, J_2, J_3 \in F$
Now $J_1 \subseteq J_2$ and $J_2 \subseteq J_3$
$\Rightarrow J_1 \subseteq J_3$
$\therefore (F, \subseteq)$ is a poset
Recall:- Theorem 1.3
By the above recall, $\exists$ a maximal ideal 'M' of a ring 'R' conta

-ining 'I'

Corollary:- 1.5 Every non-unit of a ring 'R' is contained in a maximal ideal.

Let us take 'R' be the ring and 'M' be a maximal ideal of a ring 'R' and 'a' be the non-unit element in the ring 'R'

Claim:- $a \in M$

Since a is a non-unit, by def $\exists$ a non-zero element $b \in R \ni$

$ab \neq 1 \;\forall\; 1 \in R$

$\Rightarrow aR \neq R \Rightarrow \langle a \rangle \neq R$

$\Rightarrow \overset{I}{\langle a \rangle} \neq R$ where $I = \langle a \rangle$

Now we have to show that 'I' is an ideal in a ring 'R':-

Let us take $x, y \in I$

ie, $x = ar_1$ & $y = ar_2$

where $r_1, r_2 \in R$

Now $x - y = ar_1 - ar_2 = a(r_1 - r_2) \in aR$

$\Rightarrow x - y \in aR$

$\Rightarrow x - y \in I$

Now $x \in I$ & $r_1 \in R$

$x = ar_1$ where $r_1 \in R$

consider, $xr = (ar_1)r = a(r_1 r)$

$\in aR$

$\Rightarrow xr \in I$

$\therefore$ I is an ideal in a ring 'R' and $I \neq R$

$\therefore$ I is a proper ideal of a ring 'R'

Recall:- Corollary 1.4

By the above recall $\exists$ a maximal ideal 'M' of a ring 'R' contain-ing I ie, $M \supseteq I$ (or) $I \subseteq M$

Let us take $a \in R$ be an arbitrary element

Now $a = a \cdot 1 \in aR = I \subseteq M$

$\Rightarrow a \in M$

Local ring:- A ring 'A' with exactly one-maximal ideal is called "Local ring".

Note:- 1. 'A' is a local ring if the non-units of 'A' form an idea

Residue field:- Let 'A' be the local ring and 'M' be its maximal ideal then $\frac{A}{M}$ is the "residue field".

Semi-Local ring:- A ring with finite number of maximal ideals is called semi Local ring.

Theorem:-1.6 i, Let 'A' be the ring and $M \neq \langle 1 \rangle$ be an ideal of a ring 'A' $\ni$ each element $x \in A-M$ is a unit in a ring 'A' then 'A' is a local ring and 'M' is its maximal ideal.

ii, 'A' be the ring and 'M' be maximal ideal of a ring 'A' $\ni$ every element of $1+M$ (ie, $1+x$ where $x \in M$) is a unit in a ring 'A' then 'A' is a local ring.

Proof:- i, Let 'A' be the ring and $M \neq \langle 1 \rangle$ be an ideal of a ring 'A'

ie, M is a proper ideal of a ring 'A'

Also given that $x \in A-M$ be a unit in a ring 'A'

$\Rightarrow x \in A$ and $x \notin M$

Let us take 'J' be an ideal of a ring 'A' $\ni M \subseteq J \subseteq A$ and also take $M \neq J$

To show that 'M' is maximal, it is enough to prove that $J = A$

Already, we have $J \subseteq A$

Now $x \in A$ and $x \notin M$ where $M \neq J$

$\Rightarrow x \in J$

$\forall x \in A$ we get $x \in J$

$\therefore A \subseteq J$

$\therefore A = J$

$\therefore M$ is a maximal ideal in a ring 'A'.

Uniqueness:- Let us take M_1, M_2 be two maximal ideals of a ring.

Now M_1 be maximal ideal and M_2 be ideal of a ring 'A' then $M_2 \subseteq M_1$

Now M_2 is a maximal ideal and M_1 is an ideal of a ring 'A' $M_1 \subseteq M_2$

$\therefore M_1 = M_2$

$\therefore$ The ring 'A' has a unique maximal ideal.

$\therefore$ 'A' is a local ring.

1, Given that 'A' be the ring and 'M' be the maximal ideal of a ring 'A'

Also given that every element of $1+M$ is a unit in a ring 'A'

Prove that 'A' is a local ring.

i.e., set of all non-units elements of a ring 'A' form an ideal in a ring 'A'.

for that, first we have to prove that if $a \notin M$ then 'a' is a unit element in the ring 'A'.

Now we have $a \notin M$ and M is a maximal ideal in a ring 'A' then

$M + \langle a \rangle = A$

$\Rightarrow M + aA = A$

$\Rightarrow m + ab = 1 \ \forall \ 1 \in A, m \in M \ \& \ b \in A$

$\Rightarrow ab = 1 - m$

$\Rightarrow ab = 1 + (-m) \in 1 + M$

$\Rightarrow ab \in 1 + M$

By the hypothesis, every element of $1+M$ is a unit in a ring 'A'

$\Rightarrow ab$ is a unit in a ring 'A'.

By definition, $\exists \ c \in A \ \ni (ab)c = 1$

$\Rightarrow a(bc) = 1$

$\Rightarrow$ a is a unit element in a ring 'A'.

$\therefore$ We have $a \notin M$ then 'a' is a unit element in a ring 'A'.

Illy if $a \in M$ then 'a' is a non-unit element in a ring 'A'.

$\therefore$ M be the set of all the non-unit elements of a ring 'A' and also form an ideal in the ring 'A'.

$\therefore$ 'A' is a local ring.

Nil radical:- The set 'N' of all nilpotent elements of a ring 'R' is called Nilradical of a ring 'R'. It is denoted by N(R). (Prime ideal)

Jacobson radical:- Jacobson radical of the ring 'R' is the intersection of all maximal ideals of a ring 'R' it is denoted by J(R)

Note:- 1, N(R) is the intersection of all prime ideals of a ring 'R'

2, In a principal ideal domain every non-zero prime ideal is maximal.

3, $N(R) \subseteq J(R)$ (It is equal in polynomial ring)

Theorem:- 1.7 The Set 'N' of all nilpotent elements in a ring 'A' is an ideal and $\frac{A}{N}$ has no nilpotent element $\neq 0$

Given that 'A' be the ring.

Let us take $N = \{a \in A \mid a^n = 0 \ \forall n > 0\}$

prove that 'N' is an ideal of a ring 'A'.

Let us take $a, b \in N \Rightarrow a^n = 0$ & $b^m = 0 \ \forall m, n > 0$

consider $(a+b)^{m+n} = \sum_{r=0}^{m+n} {}^{m+n}c_r \, a^{(m+n)-r} \, b^r = 0$

$\therefore (a+b)^{m+n} = 0$

$\Rightarrow a+b \in N$

Ii, Let $a \in N$ and $r \in A$

$\Rightarrow a^n = 0$

consider $(ar)^n = a^n r^n$

$\therefore (ar)^n = 0 \quad = 0 \cdot r^n = 0$

$\Rightarrow ar \in N$

$\therefore N$ is an ideal of a ring 'A'.

Now we have to prove that $\frac{A}{N}$ has no nilpotent element $\neq 0$

ie, prove that $\frac{A}{N}$ has nilpotent element $= 0$

Let us take $x+N \in \frac{A}{N}$ be the nilpotent element.

By definition $\exists \ n > 0 \ni (x+N)^n = 0+N$

$\Rightarrow x^n + N = 0+N$

$\Rightarrow x^n + N = N$

$\Rightarrow x^n \in N$

$\Rightarrow x^n$ is a nilpotent element

By definition $\exists \ m > 0 \ni (x^n)^m = 0$

$\Rightarrow x^{mn} = 0$

for all $m, n > 0 \Rightarrow nm > 0$

$\therefore \exists \ nm > 0 \ni x^{mn} = 0$

$\Rightarrow x \in N$

$\Rightarrow x+N = N = 0+N$

$\Rightarrow x+N = 0+N$

$\therefore \frac{A}{N}$ has nilpotent element $= 0$

lly $\frac{A}{N}$ has no nilpotent element $\neq 0$

Hence proved.

Theorem:- 1.8 The nilradical of a ring 'R' is the intersection of all prime ideals of a ring 'R'.

Proof:- Let us take $N(R)$ be the nilradical of a ring 'R'

We have to show that $N(R) = \cap P_i$ where P_i is a prime ideal of a ring 'R'.

ie, prove that i, $N(R) \subseteq \cap P_i$

ii, $\cap P_i \subseteq N(R)$

i, Prove that $N(R) \subseteq \cap P_i$:-

Let us take $a \in N(R)$

ie, a is a nilpotent element

By definition, $\exists$ a +ve integer $n > 0 \ni a^n = 0$

And also take $0 \in P_i$ where P_i is a prime ideal in a ring 'R'.

$\Rightarrow a^n \in P_i$

$\Rightarrow a \times a^{n-1} \in P_i$

$\because P_i$ is the prime ideal, by definition either $a \in P_i$ or $a^{n-1} \in P_i$

Let us take $a \in P_i$ for each (i)

$\Rightarrow a \in \cap P_i$

$\therefore N(R) \subseteq \cap P_i$ —(i)

ii, Prove that $\cap P_i \subseteq N(R)$:-

Let us take $a \in \cap P_i$ where P_i is a prime ideal in 'R'.

$\Rightarrow a \in P_i$ for each i

Prove that $a \in N(R)$

Prove that 'a' is a nilpotent element in the ring 'R'.

ie, prove that 'a' is not the nilpotent element

Suppose on the contrary, 'a' is not the nilpotent element

By def, $\exists$ 'n' $> 0 \ni a^n \neq 0$

consider the set, $S = \{a^n / n > 0\}$

Let us take the set $\Sigma = \{I / I$ is an ideal in a ring 'R' and $I \cap S = \phi\}$

Define a relation '$\subseteq$' on Σ

Then clearly $(\Sigma, \subseteq)$ is a poset

Let us take $\{I_\alpha / \alpha \in \Delta\}$ be a chain in a poset $(\Sigma, \subseteq)$

Let us take $J = \bigcup_{\alpha \in \Delta} I_\alpha$

clearly, J is an ideal of a ring 'R' and $J \cap S = (\cup I_\alpha) \cap (S)$

$= \cup (I_\alpha \cap S)$

$= \cup \phi$

$= \phi$

$\therefore$ J is an ideal of a ring 'R' and $J \cap S = \phi$

$\therefore$ J is an upper bound for the chain in a poset.

By zorn's lemma, the poset has a maximal element say 'p'.

Claim:- 'p' is a prime ideal.

Suppose on the contrary, p is not a prime ideal.

By definition, for all $x, y \in R \ni xy \in p \Rightarrow x \notin p$ and $y \notin p$

$\therefore \exists$ two proper ideals $p + \langle x \rangle$ and $p + \langle y \rangle$, both are containing 'p'

ie, $p \subseteq p + \langle x \rangle$ and $p \subseteq p + \langle y \rangle$

$\Rightarrow p + \langle x \rangle \notin \Sigma$ and $p + \langle y \rangle \notin \Sigma$ ($\because \Sigma$ has a maximal element 'p' but here $p+\langle x \rangle$ is a maximal element so it is a contradiction)

$\Rightarrow [p + \langle x \rangle] \cap S \neq \phi$ and $[p + \langle y \rangle] \cap S \neq \phi$

Let us take $a^m \in [p + \langle x \rangle] \cap S$ and $a^n \in [p + \langle y \rangle] \cap S$

$\Rightarrow a^m \in p + \langle x \rangle$ and $a^m \in S$ and $a^n \in p + \langle y \rangle$ and $a^n \in S$

Now $a^m \in p + \langle x \rangle$ and $a^n \in p + \langle y \rangle$

$a^m = p_1 + x l_1$ and $a^n = p_2 + y l_2$ where $p_1, p_2 \in P$ & $l_1, l_2 \in R$

Now $a^m \cdot a^n = (p_1 + x l_1) \cdot (p_2 + y l_2)$

$$= p_1 p_2 + p_1 y l_2 + p_2 x l_1 + x y l_1 l_2 \in P$$ (by def of ideal (ii) condition $a \in p, a \in p, a \in p$.)

$a^{m+n} \in P$

Also we have $a^m \in S$ and $a^n \in S$

$\Rightarrow a^m \cdot a^n \in S$

$\Rightarrow a^{m+n} \in S$

$\therefore$ we have $a^{m+n} \in p$ and $a^{m+n} \in S$

$\Rightarrow a^{m+n} \in p \cap S$

$\therefore p \cap S \neq \phi$

$\Rightarrow p \notin \Sigma$

which is a contradiction to Σ has a maximal element 'p'.

$\therefore$ Our assumption is wrong.

$\therefore$ p is a prime ideal.

Now we have $a \in P_i$ & $a^n \in S$ for some $n > 0$

$\Rightarrow a \in S$ for $1 > 0$ (by taking n=1)

$\Rightarrow a \in P_i \cap S = \phi \Rightarrow a \in \phi$

$\Rightarrow a = 0$

$\Rightarrow$ 'a' is a nilpotent element, which is a contradiction.

$\therefore$ Our assumption is wrong.

$\therefore$ 'a' is a nilpotent element.

ie, $a \in N(R)$

$\therefore \cap p_i \subseteq N(R)$ — (2)

From (1) and (2) $N(R) = \cap p_i$

where p_i is a prime ideal of a ring 'R'.

Hence proved.

***Theorem:- 1.9** $a \in J(R) \Leftrightarrow 1+ab$ is a unit in a Ring 'R' $\forall b \in R$

(or)

$a \in J(R) \Leftrightarrow 1-ab$ is a unit in a ring 'R' $\forall b \in R$

Proof:- Let $J(R)$ be the Jacobson radical of a ring 'R'

By definition, $J(R)$ is the intersection of all maximal ideals in a ring 'R'.

Let us take $a \in J(R)$

$\Rightarrow a \in \cap M$ where M is a maximal ideal

$\Rightarrow a \in M$, for every maximal ideal 'M'.

Prove that $1+ab$ is a unit in a ring 'R'.

Suppose on the contrary, $1+ab$ is not a unit in a ring 'R'.

$\Rightarrow 1+ab \in M$ ($\because$ corollary - 1.5)

we have $a \in M$ and M is an ideal in 'R'.

By def, $\forall b \in R \ni ab \in M$

$\therefore$ we have $1+ab, ab \in M$ and M is an ideal in 'R'

$\Rightarrow 1+ab-ab \in M$

$\Rightarrow 1 \in M$ and M is an ideal in a ring 'R'.

$\Rightarrow M = R$

which is a contradiction to 'M' is maximal ideal.

$\therefore$ Our assumption is wrong

$\therefore 1+ab$ is a unit in a ring 'R' $\forall b \in R$

Conversely, suppose that $1+ab$ is a unit in a ring 'R'.

Prove that $a \in J(R)$

Suppose on the contrary, $a \notin J(R)$

$\Rightarrow a \notin \cap M$ where 'M' is a maximal ideal in a ring 'R'

$\Rightarrow a \notin M$ for every maximal ideal 'M' in a ring 'R'.

$\therefore$ By the known theorem, we have $M + \langle a \rangle = R$

$\Rightarrow M + aR = R$

$\Rightarrow m + ab = 1 \ \forall m \in M, a, b, 1 \in R$

$\Rightarrow m = 1 - ab$

$\Rightarrow m = 1 + a(-b)$,

∵ By the hypothesis, we have ...

∵ $\Rightarrow 1+a(-b)$ is a unit in a ring 'R'.

B $\Rightarrow m$ is a unit in a ring 'R'.

C By definition $\exists\, n \in R \ni mn = 1 = nm$

S Now let us take $r \in R$ be an arbitrary element.

B, Now $r = 1 \cdot r = (mn) \cdot r$

$= m(nr) \in M$

(∵ $n, r \in R$, $m \in M$ by def of ideal ⓘ and $nr \in R$, $m \in M$ $m(nr) \in M$)

∴ $r \in M$

ie ∴ $\forall\, r \in R$, we get $r \in M \Rightarrow R \subseteq M$

$\Rightarrow$ But we have M is an ideal in a ring 'R' $\Rightarrow M \subseteq R$

$\Rightarrow$ ∴ $M = R$

Le which is a contradiction to 'M' is maximal ideal.

$\Rightarrow$ ∴ Our assumption is wrong.

N ∴ $a \in J(R)$

Note:- * Let I, J are ideals in a ring 'R' then

N 1, $I \cap J$ is also ideal in a ring 'R'.

2, $I \cup J$ is an ideal in a ring 'R' $\Leftrightarrow I \subseteq J$ (or) $J \subseteq I$

3, $I + J = \{\Sigma(a_i + b_i) \mid a_i \in I, b_i \in J\}$ is also an ideal in a ring 'R'

All 4, $I \cdot J = \{\Sigma(a_i \cdot b_i) \mid a_i \in I, b_i \in J\}$ is also an ideal in a ring 'R'

$\Rightarrow$ 5, $I \cdot J \subseteq I \cap J$

$\Rightarrow$ 6, $I \cdot J \neq I \cap J$

∴ 7, $I \cdot J = I \cap J$ only if I, J are co-prime (or) co-maximal (or) $I+J=1$

* Let 'A' be the ring and $I_1, I_2, \ldots I_n$ be ideals in the ring then define a mapping $f: A \to \prod_{i=1}^{n} \frac{A}{I_i}$ given by $f(x) = (x+I_1, x+I_2, x+I_3 \ldots x+I_n)$

∴ Proposition:- 1.10 i, If I_i, I_j are co-prime whenever $i \neq j$ then $\prod I_i =$ ∴ $\bigcap_{i=1}^{n} I_i$

N ii, 'f' is surjective $\Leftrightarrow I_i, I_j$ are co-prime whenever $i \neq j$

iii, f is injective $\Leftrightarrow \bigcap_{i=1}^{n} I_i = \langle 0 \rangle$

Proof:- Given that I_i, I_j are co-prime whenever $i \neq j$

$\Rightarrow$ Let us take $I_1, I_2 \ldots I_n$ are co-maximal whenever $i \neq j$

∴ claim: it, $\prod_{i=1}^{n} I_i = \bigcap_{i=1}^{n} I_i$

∴ This is proved by mathematical induction on 'i'

Suppose $i=1$ then the result is trivally true.

Now take $n=2$ then we have to show that $I_1 \cdot I_2 = I_1 \cap I_2$

ie, prove that $I_1 \cdot I_2 \subseteq I_1 \cap I_2$ and $I_1 \cap I_2 \subseteq I_1 \cdot I_2$

Let us take an element $x \in I_1 \cap I_2$

$\Rightarrow x \in I_1$ and $x \in I_2$

Since I_1, I_2 are co-maximal, by definition $I_1 + I_2 = R$

$\Rightarrow a + b = 1$ where $a \in I_1, b \in I_2$ and $1 \in R$

Now $x = x \cdot 1$

$= x(a+b)$

$= xa + xb \in I_1 \cdot I_2$

$x \in I_1 \cdot I_2$

$\therefore \forall x \in I_1 \cap I_2$ we get $x \in I_1 \cdot I_2$

$\therefore I_1 \cap I_2 \subseteq I_1 \cdot I_2$

Let us take an element $x \in I_1 \cdot I_2$

$\Rightarrow x \in \sum a_i \cdot b_i$ where $a_i \in I_1, b_i \in I_2$

$\Rightarrow \sum a_i b_i \in I_1 \cap I_2$ ($a_i \in I_1, b_i \in I_2$, $I_1, I_2 \subseteq R$ $b_i \in R$ $a_i \in I$ $b_i \in R \Rightarrow a_i b_i \in I_1$)

$\therefore I_1 \cdot I_2 \subseteq I_1 \cap I_2$

$\therefore I_1 \cdot I_2 = I_1 \cap I_2$

$\therefore$ The result is also true for $i=2$ $a_i \in I_2$ $b_i \in R \Rightarrow a_i b_i \in I_2$

Now assume that the result is true for $i = n-1$ ie, $\bigcap_{i=1}^{n-1} I_i = \prod_{i=1}^{n-1} I_i$

Now we have to prove that result is also true for 'n'.

Let us take $J = \bigcap_{i=1}^{n-1} I_i$

clearly, J is an ideal in a ring 'R'. And also we have I_n is an ideal in a ring 'R'. Then $J + I_n$ is an ideal in a ring 'R'.

ie, $J + I_n \subseteq R$ —(1)

Now we have to show that $R \subseteq J + I_n$

we have $I_1, I_2 \dots I_n$ are co-maximal.

ie, $I_i + I_j = R$ where $i \neq j$

$\Rightarrow a_i + b_j = 1$ where $a_i \in I_i$ & $b_j \in I_j$, $1 \in R$

$\Rightarrow a_i = 1 - b_j$

Take an element $a = \prod_{i=1}^{n-1} a_i = \prod_{i=1}^{n-1} (1 - b_j)$ where $b_j \in I_n$

$= 1 - b$ where $b \in I_n$ & $a_i \in I_i$ $\forall i = 1, 2, 3 \dots n-1$ → only option n

$\Rightarrow a \in J$

$a = \prod_{i=1}^{n-1} a_i \in \prod_{i=1}^{n-1} I_i = \bigcap_{i=1}^{n-1} I_i = J$

$\therefore a = 1 - b$

$\Rightarrow a + b = 1$

ie, $1 = a + b \in J + I_n$

$\therefore 1 \in J + I_n$

$\Rightarrow R \subseteq J + I_n$ — (2) From (1) and (2)

J, I_n are co-maximal.

$\therefore J \cdot I_n = J \cap I_n$

$\left(\bigcap_{i=1}^{n-1} I_i\right) \cdot I_n = \left(\bigcap_{i=1}^{n-1} I_i\right) \cap I_n = \bigcap_{i=1}^{n-1} (I_i \cap I_n)$

$\Rightarrow \left(\prod_{i=1}^{n-1} I_i\right) I_n = \bigcap_{i=1}^{n} I_i \qquad = \bigcap_{i=1}^{n} I_i$

$\Rightarrow \prod_{i=1}^{n} I_i = \bigcap_{i=1}^{n} I_i$

ii, Now define a mapping $f: R \rightarrow \prod_{i=1}^{n} \frac{R}{I_i}$ is given by $f(x) = (x+I_1, x+I_2, ---- x+I_n \ \forall x \in R)$

where $I_1, I_2, I_3 --- I_n$ are ideals in a ring 'R'.

Now let us take 'f' is surjective

Prove that I_i, I_j are co-maximal whenever $i \neq j$

ie, Prove that $I_i + I_j = R$

Always we have $I_i + I_j \subseteq R$ —(3)

Now, it is enough to prove that $R \subseteq I_i + I_j$

consider the element $(\bar{0}, \bar{0}, --- \bar{1}, --- \bar{0})$ in the i^{th} place is $\bar{1}$ (8) $\bar{1}$ in the i^{th} place.

Since f is onto, by definition for every $(\bar{0}, \bar{0}, --- \bar{1}, --- \bar{0}) \in \prod_{i=1} \frac{R}{I_i}$, ∃ an element $a \in R \ni f(a) = (\bar{0}, \bar{0} --- \bar{1}, --- \bar{0})$

$(a+I_1, a+I_2 --- a+I_n) = (0+I_1, 0+I_2, --- 1+I_i, --- 0+I_n)$

Equating the terms on both sides, we get

$a+I_1 = 0+I_1, \ a+I_2 = 0+I_2, ---- a+I_i = 1+I_i, ---- a+I_n = 0+I_n$

$\Rightarrow a+I_1 = I_1, \ a+I_2 = I_2, ----- a+I_i = 1+I_i, ---- a+I_n = I_n$

$\Rightarrow a \in I_1, a \in I_2, -- a \notin I_i, ---- a \in I_n$

ie, $a \in I_j \ \forall j = 1, 2, 3 --- n$ where $j \neq i$.

and $1-a \in I_i$

$\Rightarrow (1-a) + a \in I_i + I_j$ whenever $j \neq i$

$\Rightarrow 1 \in I_i + I_j$ whenever $i \neq j$

$\Rightarrow R \subseteq I_i + I_j$ —(4)

From (3) and (4), $R = I_i + I_j$ whenever $i \neq j$

$\therefore I_i, I_j$ are co-prime.

Conversely, suppose that I_i, I_j are co-maximal whenever $i \neq j$

ie, $I_i + I_j = R$ whenever $i \neq j$

Prove that 'f' is surjective.

we have $I_i + I_j = R$

$\Rightarrow u_i + v_j = 1$ where $u_i \in I_i$ & $v_j \in I_j$ and $1 \in R$

consider, an element $x = \prod v_j, j = 2,3 \dots n$

$$= \prod_{j=2}^{n} (1 - u_i) = 1 + u$$

$\Rightarrow x = 1 + u$

$\Rightarrow x - 1 = u \in I_i$ (or) I_1

$\Rightarrow x - 1 \in I_1$

$\Rightarrow x + I_1 = 1 + I_1$

Also we have $j = 2,3 \dots n$

$\therefore$ we have $x = \prod_{j=2}^{n} v_j$

$\Rightarrow x \in \prod_{j=2}^{n} I_j$

$\Rightarrow x \in \bigcap_{j=2}^{n} I_j$

$\Rightarrow x \in (I_2 \cap I_3 \cap \dots \cap I_n)$

$\Rightarrow x \in I_2, x \in I_3, \dots x \in I_n$

$\Rightarrow x + I_2 = I_2, x + I_3 = I_3 \dots x + I_n = I_n$

$\Rightarrow x + I_2 = 0 + I_2, x + I_3 = 0 + I_3 \dots x + I_n = 0 + I_n$

$\Rightarrow x + I_1, x + I_2, x + I_3, x + I_4 \dots x + I_n = 1 + I_1, 0 + I_2, 0 + I_3, \dots 0 + I_n$

$\Rightarrow f(x) = (\bar{1}, \bar{0}, \bar{0} \dots \bar{0})$

$\therefore f$ is onto.

iii) Let us take 'f' is injective.

Prove that $\bigcap_{i=1}^{n} I_i = \langle 0 \rangle$

we know that the mapping 'f' is one-one $\Leftrightarrow \ker f = \langle 0 \rangle$

Now Kernal $f = \{x \in R \mid f(x) = (\bar{0}, \bar{0}, \dots \bar{0}, \dots \bar{0})\}$

$$= \{x \in R \mid ((x + I_1), x + I_2, x + I_3 \dots (x + I_n)) = (\bar{0}, \bar{0}, \bar{0} \dots \bar{0})\}$$

$$= \{x \in R \mid x + I_1 = 0 + I_1, x + I_2 = 0 + I_2, \dots x + I_n = 0 + I_n\}$$

$$= \{x \in R \mid x + I_1 = I_1, x + I_2 = I_2 \dots x + I_n = I_n\}$$

$$= \{x \in R \mid x \in I_1, x \in I_2, x \in I_3 \dots x \in I_n\}$$

$$= \{x \in R \mid x \in \bigcap_{i=1}^{n} I_i\}$$

$$\ker f = \bigcap_{i=1}^{n} I_i$$

Since f is one-one, we have $\ker f = 0$

$$\Rightarrow \left(\ker f = \bigcap_{i=1}^{n} I_i\right) \text{ ie, } \bigcap_{i=1}^{n} I_i = 0$$

Conversely, suppose that $\bigcap_{i=1}^{n} I_i = \langle 0 \rangle$

Let us take $x, y \in A \ni f(x) = f(y)$

$\Rightarrow (x+I_1, x+I_2 \dots x+I_n) = (y+I_1, y+I_2, y+I_3 \dots y+I_n)$

$\Rightarrow x+I_1 = y+I_1, x+I_2 = y+I_2, \dots x+I_n = y+I_n$

$\Rightarrow x-y \in I_1, x-y \in I_2 \dots x-y \in I_n$

$\Rightarrow x-y \in I_1 \cap I_2 \cap \dots I_n$

$\Rightarrow x-y \in \bigcap_{i=1}^{n} I_i$

$\Rightarrow x-y = \langle 0 \rangle$

$\Rightarrow x-y = 0 \Rightarrow x = y$

$\therefore f$ is injective or one-one.

Proposition:- 1.11

i, Let $P_1, P_2 \dots P_n$ be prime ideals and 'I' be an ideal contain in $\bigcup_{i=1}^{n} P_i$ then $I \subseteq P_i$ for some 'i'

ii, Let $I_1, I_2, I_3 \dots I_n$ be ideals and 'p' be the prime ideal containing intersection $\bigcap_{i=1}^{n} I_i$ then $P \supseteq I_i \ \forall i$

iii, If $P = \bigcap_{i=1}^{n} I_i$ then $P = I_i$ for some 'i'

Proof:- Given that $P_1, P_2, P_3 \dots P_n$ be the prime ideals and 'I' be an ideal in a ring 'R' $\ni I \subseteq \bigcup_{i=1}^{n} P_i$

Claim:- $I \subseteq P_i$ for some 'i'

This is proved by mathematical induction on 'i'

Suppose $i=1$ then $I \subseteq P_1$

$\therefore$ The result is trivaly true.

Suppose $i=2$ then $I \subseteq P_1 \cup P_2$

$\Rightarrow I \subseteq P_1$ or $I \subseteq P_2$

$\therefore$ The statement is also true for $i=2$

Now assume that statement is true upto $n-1$

ie, $I \subset \bigcup_{i=1}^{n-1} P_i \Rightarrow I \subseteq P_i$ for some 'i'

Now we have to show that result is also true for 'n'.

Let us take $I \subseteq \bigcup_{i=1}^{n} P_i$

$$I \subseteq \left(\bigcup_{i=1}^{n-1} P_i\right) \cup P_n \quad (\text{by } i=2)$$

$$I \subseteq \bigcup_{i=1}^{n-1} P_i \text{ or } I \subseteq P_n$$

If $I \subseteq \bigcup_{i=1}^{n-1} P_i$ then by induction hypothesis the result is true upto ie, $I \subseteq P_i$ for some 'i'

If not ie, $I \notin \bigcup_{i=1}^{n-1} P_i$

Let us take an element $a_i \in I$ and $a_i \notin P_j$ where $j \neq i, 1 \leq i \leq n$

If for some i, also we have $a_i \notin P_i$

$\Rightarrow a_i \notin \bigcup_{i=1}^{n} P_i$

$\Rightarrow I \notin \bigcup_{i=1}^{n} P_i$

which is a contradiction

$\therefore$ Our assumption is wrong.

$\therefore$ for some 'i' we must have $a_i \in P_i$

$\therefore I \subseteq P_i$ for some 'i'

Let us take an element $a = \sum_{i=1}^{n} a_1 \cdot a_2 \cdot a_3 \cdots a_{i-1} \cdot a_{i+1} \cdots a_n \in I$

but the element 'a' is not in $\bigcup_{i=1}^{n} P_i$, which is a contradiction.

$\therefore$ we have $I \subseteq \bigcup_{i=1}^{n} P_i \Rightarrow P_i$ for some 'i'

ii, Given that $I_1, I_2, I_3 \cdots I_n$ be the ideals and 'p' be the prime ideal in a ring 'R' $\ni P \supseteq \bigcap_{i=1}^{n} I_i$

claim:- $P \supseteq I_i \; \forall i$

Suppose on the contrary, $P \not\supseteq I_i$

ie, $P \subseteq I_i$

Now choose an element $a_i \in I_i$ & $a_i \notin P$ for $i = 1, 2, 3 \cdots n$

Now consider an element $a = \prod_{i=1}^{n} a_i \in \prod_{i=1}^{n} I_i = \bigcap_{i=1}^{n} I_i \subseteq P$ (by previous thm :))

$\Rightarrow a = \prod_{i=1}^{n} a_i \in P$

Since 'p' is a prime ideal. By definition, we have $a_i \in P$

which is a contradiction to $a_i \notin P$

$\therefore$ Our assumption is wrong.

$\therefore P \supseteq I_i$

iii, Given that $P = \bigcap_{i=1}^{n} I_i$

$\Rightarrow P \subseteq \bigcap_{i=1}^{n} I_i$ and $P \supseteq \bigcap_{i=1}^{n} I_i$

$\Rightarrow P \subseteq I_i$ and $P \supseteq I_i$

$\Rightarrow P = I_i \; \forall i = 1, 2, 3 \cdots n$

Definition:- If I, J are ideals in a ring 'R' then the ideal quo-ent is $(I:J) = \{x \in R \mid xJ \subseteq I\}$ is also an ideal in the ring

Let us take $I = \langle 0 \rangle$ then $(0:J) = \{x \in R \mid xJ \subseteq \langle 0 \rangle\}$

ie, $xJ \subseteq 0 \Rightarrow xJ = \langle 0 \rangle$

ie, $xy = 0 \ \forall \ y \in J$ which is called "Annhilator Ideal" of J and it is denoted by Ann J

Note:- for any $x \in \text{Ann} J$ then $xy = 0 \ \forall \ y \in J$

Radical of an ideal:- If 'I' is an ideal in a ring 'R' then radical of an ideal 'I' is $r(I) = \{x \in R \mid x^n \in I$ for some $n > 0\}$ is also an ideal in a ring 'R'.

Note:- 'D' be the set of all zero-divisors of a ring 'R' then $D = \bigcup_{x \in R} \text{Annh}(x)$

Proposition:- 1.12 Let 'A' be the ring and I, J, k be ideals of a ring 'A' then prove that i, $I \subseteq (I:J)$

ii, $(I:J)J \subseteq I$

iii, $((I:J):K) = (I:JK) = ((I:K):J)$

iv, $\left(\bigcap_{i=1}^{n} I_i : J\right) = \bigcap_{i=1}^{n} (I_i : J)$

v, $\left(I : \sum_{i=1}^{n} J_i\right) = \bigcap_{i=1}^{n} (I : J_i)$

Proof:- Given that 'A' be the ring and I, J, K are ideals of a ring 'A'

i, Let us take an element $x \in I$

$\Rightarrow x \cdot 1 \in I$

$\Rightarrow xA \subseteq I \ \forall \ 1 \in A$

$\Rightarrow x \in (I:A)$

$\Rightarrow x \in (I:J)$ for some $J \subseteq A$

$\therefore I \subset (I:J)$

ii, Let us take an element $x \in (I:J)$

$\Rightarrow xJ \subseteq I$

$\Rightarrow (I:J)J \subseteq I$

iii, Let us take an element $x \in (I:J):K$

$\Leftrightarrow xk \subseteq (I:J)$

$\Leftrightarrow xkj \subseteq I$

$\Leftrightarrow x \in (I:(J K))$ $\therefore (I:J):K = (I:JK)$

Let us take an element $x \in (I:K):J$

$\Leftrightarrow xj \subseteq (I:K)$

$\Leftrightarrow xjk \subseteq I$

$\Leftrightarrow x \in (I : JK) \quad \because (I:K):J = (I:JK)$

$\therefore (I:J):K = (I:JK) = (I:K):J$

iv, Let us take an element $x \in (\bigcap_{i=1}^{n} I_i : J)$

$\Leftrightarrow xj \subseteq \bigcap_{i=1}^{n} I_i$

$\Leftrightarrow xj \subseteq I_i$ for every $i = 1,2,3 \dots n$

$\Leftrightarrow x \in (I_i : J) \; \forall i = 1,2,3 \dots n$

$\Leftrightarrow x_i \in \bigcap_{i=1}^{n} (I_i : J)$

v, Let us take $x \in (I : \sum_{i=1}^{n} J_i)$

$\Leftrightarrow x \sum_{i=1}^{n} J_i \subseteq I$

$\Leftrightarrow \sum_{i=1}^{n} x J_i \subseteq I$

$\Leftrightarrow xJ_i \subseteq I \; \forall i = 1,2,3 \dots n$

$\Leftrightarrow x \in (I : J_i) \; \forall i = 1,2,3 \dots n$

$\Leftrightarrow x \in \bigcap_{i=1}^{n} (I : J_i)$

Proposition 1.13:- If I, J are ideals in a ring 'A' then

i, $I \subseteq r(I)$

ii, $r[r(I)] = r(I)$

iii, $\overset{1}{r(IJ)} = \overset{2}{r(I \cap J)} = \overset{3}{r(I) \cap r(J)}$ $\quad 1=3 \quad 2=3 \Rightarrow 1=2=3$

iv, $r(I) = \langle 1 \rangle \Leftrightarrow I = \langle 1 \rangle$

v, $r(I+J) = r[r(I) + r(J)]$

vi, If P is a prime ideal then $r(P^n) = P$ for some $n > 0$

Proof:- Given that 'A' be the ring and I, J are ideals of ring 'A'

i, Let us take $x \in I$

$\Rightarrow x^1 \in I$

$\Rightarrow x \in r(I)$

$\therefore I \subseteq r(I)$

ii, prove that $r[r(I)] = r(I)$

ie, prove that $r[r(I)] \subseteq r(I)$ & $r(I) \subseteq r[r(I)]$

we know that $I \subseteq r(I)$

$r(I) \subseteq r[r(I)]$ — (a)

Now let us take an element $x \in r[r(I)]$

$\Rightarrow x^n \in r(I)$ for some $n > 0$

$\Rightarrow (x^n)^m \in I$ for some $m > 0$

$\Rightarrow x^{nm} \in I$ for some $nm > 0$

$\Rightarrow x \in r(I)$

$r[r(I)] \subseteq r(I)$ — (b)

From (a) and (b) $r[r(I)] = r(I)$

ii, prove that $2 = 3$

ie, prove that $r(I \cap J) = r(I) \cap r(J)$

we know that $I \subseteq J \Rightarrow r(I) \subseteq r(J)$

we have $I \cap J \subseteq I$ and $I \cap J \subseteq J$

$\Rightarrow r(I \cap J) \subseteq r(I)$ $\qquad$ $\Rightarrow r(I \cap J) \subseteq r(J)$

$\cancel{\Rightarrow r(I) \cap r(J) \subseteq r(I)}$ $\qquad$ $\Rightarrow r(I \cap J) \subseteq r(I) \cap r(J)$ — (1)

Now take an element $x \in r(I) \cap r(J)$

$\Rightarrow x \in r(I)$ and $x \in r(J)$

$\Rightarrow x^n \in I$ and $x^m \in J$ for some $n, m > 0$

$\Rightarrow x^n \cdot x^m \in I \cap J$ $(\because I \cap J \subseteq I \cap J)$

$\Rightarrow x^{n+m} \in I \cap J$ for some $n+m > 0$

$\Rightarrow x \in r(I \cap J)$

$\Rightarrow r(I) \cap r(J) \subseteq r(I \cap J)$ — (2)

From (1) and (2)

$r(I \cap J) = r(I) \cap r(J)$

using (i) condition
$x^n \in I$, $x^m \in R$
$x^n x^m \in I$
$x^m \in J$, $x^n \in R$
$x^{m \cdot n} \in J$
$x^{mn} \in (I \cap J)$

prove that $1 = 3$

ie, to prove that $r(I \cdot J) = r(I) \cap r(J)$

ie, to prove that $r(I \cdot J) \subseteq r(I) \cap r(J)$ & $r(I) \cap r(J) \subseteq r(I \cdot J)$

we know that $r(I \cdot J) \subseteq \cancel{r(I) \cap r(J)}$ $I \cap J$

$r(I \cdot J) \subseteq \cancel{r(I) \cap r(J)}$ $r(I \cap J)$ $(\because 2 = 3)$

$\Rightarrow r(I \cdot J) \subseteq r(I) \cap r(J)$ — (3)

Let us take $x \in r(I) \cap r(J)$

$\Rightarrow x \in r(I)$ and $x \in r(J)$

$\Rightarrow x^n \in I$ for some $n > 0$ & $x^m \in J$ for some $m > 0$

$\Rightarrow x^n \cdot x^m \in I \cdot J$ for some $m, n > 0$

$\Rightarrow x^{n+m} \in I \cdot J$ for some $n+m > 0$

$\Rightarrow x \in r(I J)$

$\therefore r(I) \cap r(J) \subseteq r(I \cdot J)$ — (4)

From (3) and (4) $r(I) \cap r(J) = r(I \cdot J)$

$\therefore$ we have $(1) = (3)$ & $(2) = (3)$

$\Rightarrow (1) = (2) = (3)$

$\therefore\ r(I \cdot J) = r(I) \cap r(J) = r(I \cap J)$

iv) Let us take $r(I) = \langle 1 \rangle$

$= 1 \cdot A = A$

$\Leftrightarrow 1 \in r(I)$ $\quad \cancel{r(I) = A}$

$\Leftrightarrow 1^n \in I$ for some $n > 0$ $\quad \cancel{\Leftrightarrow 1 \in r(I)}$

$\Leftrightarrow 1 \in I$ $\quad \cancel{\Leftrightarrow 1^n \in I \text{ for some } n > 0}$

$\Leftrightarrow I = A = 1 \cdot A = \langle 1 \rangle$ $\quad \cancel{\Leftrightarrow 1^{(n)} \in I}$

$\cancel{\Leftrightarrow 1 \in I}$

$\cancel{\Leftrightarrow A = I}$

$\Leftrightarrow$

v) we know that $I \subseteq I + J$ and $J \subseteq I + J$

$\Rightarrow r(I) \subseteq r(I+J) \text{ --- } (1)$ $\quad \Rightarrow r(J) \subseteq r(I+J) \text{ --- } (2)$

$(1) + (2) \Rightarrow r(I) + r(J) \subseteq r(I+J)$

$\Rightarrow r[r(I) + r(J)] \subseteq r[r(I+J)]$

$\Rightarrow r[r(I) + r(J)] \subseteq r(I+J)$ ($\because$ from ②) — (3)

conversely, we know that $I \subseteq r(I)$ & $J \subseteq r(J)$

$\Rightarrow I + J \subseteq r(I) + r(J)$

from (3) and (4) $\quad \Rightarrow r(I+J) \subseteq r(r(I) + r(J))$ — (4)

$\therefore r(I+J) = r(I) + r(J)$

vi) Given that 'p' be the prime ideal.

prove that $P \subseteq r(P^n)$ & $r(P^n) \subseteq P$

Let us take $x \in P$

$\Rightarrow x^n \in P^n$

$\Rightarrow x \in r(P^n)$

$\therefore P \subseteq r(P^n)$ — (1)

conversely, Let us take $x \in r(P^n)$

$\Rightarrow x^k \in P^n$ for some $k > 0$

ie, $x^k = y_1 \cdot y_2 \cdot y_3 \cdots y_n$ where each $y_i \in P$

$\Rightarrow x^k \in P$

$\Rightarrow x \cdot x^{k-1} \in P$

$\Rightarrow x \in P$ ($\because$ P is a prime ideal, by def, $x \in P$ or $x^{k-1} \in P$)

$\therefore r(P^n) \subseteq P$ — (2)

from (1) and (2)

$r(P^n) = P$

Hence proved.

Proposition:- 1.14 The radical of an ideal is the intersection of all prime ideals which contain the ideal 'I'.

Proof:- Given that 'R' be the ring and 'I' be an ideal in a 'R'.

$\Rightarrow r(I)$ is also an ideal in a ring 'R'.

Let us take $p : R \longrightarrow \frac{R}{I}$ be the projection mapping

$\Rightarrow p^{-1} : \frac{R}{I} \rightarrow R$ exists. (Projection mapping always continuous and inverse exists)

We know that $N(R)$ is the intersection of all prime ideals which contain an ideal I.)

$\Rightarrow N\left(\frac{R}{I}\right)$ is also intersection of all prime ideals which contain an ideal 'I'.

Claim:- $p^{-1}\left(N\left(\frac{R}{I}\right)\right) = r(I)$

Let us take $x \in R \ni x \in r(I)$

$\Leftrightarrow \exists\ n > 0 \ni x^n \in I$

$\Leftrightarrow x^n + I = I$

$\Leftrightarrow x^n + I = 0 + I$

$\Leftrightarrow x^n + I = \bar{0}$

$\Leftrightarrow p(x^n) = \bar{0}$

$\Leftrightarrow p(x \cdot x \cdot x \cdots x \text{ (n times)}) = \bar{0}$

$\Leftrightarrow p(x) \cdot p(x) \cdots p(x) \text{ (n times)} = \bar{0}$

$\Leftrightarrow [p(x)]^n = \bar{0}$

$\Leftrightarrow p(x) \in N\left(\frac{R}{I}\right)$

$\Leftrightarrow x \in p^{-1}\left(N\left(\frac{R}{I}\right)\right)$

Hence proved.

Proposition:- 1.15 Let 'D' be the set of all zero-divisors in a ring 'A' then $D = \bigcup_{x \in A} r[\text{Ann}\langle x \rangle]$

Proof:- Given that 'D' be the set of all zero-divisors in a ring 'A'.

Claim:- $D = \bigcup_{x \in A} r[\text{Ann}\langle x \rangle]$

for that, first we have to prove that 1, $D = r(D) = r\left[\bigcup_{x \in A} \text{Ann}\langle x \rangle\right]$

$$\text{ii, } r\left[\bigcup_{x \in A} \text{Ann}\langle x \rangle\right] = \bigcup_{x \in A}\left[r(\text{Ann}\langle x \rangle)\right]$$

To prove i, we know that $I \subseteq r(I)$

ie, $D \subseteq r(D)$ —(1)

Now it is enough to prove that $r(D) \subseteq D$

Let us take an element $x \in r(D)$ where $x \neq 0$

$\Rightarrow x^n \in D$ for some $n>0$

$\Rightarrow \exists\, y \in A \ni x^n(y) = 0$ ($\because$ 'D' is the set of all zero-divisors)

$\Rightarrow x(x^{n-1}.y) = 0$

If $x^{n-1}.y \neq 0$ then $x \in D$

Then the proof is over.

Suppose $x^{n-1}.y = 0$

$\Rightarrow x(x^{n-2}.y) = 0$

if $x(x^{n-2}y) \neq 0$ then $x \in D$ if not ie, $x^{n-2}.y = 0$

$\Rightarrow x(x^{n-3}.y) = 0$

continuing this process after $(n-1)$ steps we have either $x \in D$ or

$xy = 0$

Hence $x \in D$

$\therefore r(D) \subseteq D$ —(2)

From (1) & (2) $r(D) = D$

$$r\left[\bigcup_{x \in A} Ann\langle x\rangle\right] = D$$

To prove ii :—

Prove that $r\left[\bigcup_{x \in A} Ann\langle x\rangle\right] = \bigcup_{x \in A}\left[r(Ann\langle x\rangle)\right]$

Always we have $\bigcup_{x \in A}\left[r(Ann\langle x\rangle)\right] \subseteq r\left[\bigcup_{x \in A} Ann\langle x\rangle\right]$ —(1)

Now it is enough to prove that $r\left[\bigcup_{x \in A} Ann\langle x\rangle\right] \subseteq \bigcup_{x \in A}\left[r(Ann\langle x\rangle)\right]$

Let us take an element $x \in r\left[\bigcup_{x \in A} Ann\langle x\rangle\right]$

$\Rightarrow x^n \in \bigcup_{x \in A} Ann\langle x\rangle$ for some $n>0$

$\Rightarrow x^n \in Ann\langle x\rangle$ for some $x \in A$

$\Rightarrow x \in r[Ann\langle x\rangle]$ for some $x \in A$

$\Rightarrow x \in \bigcup_{x \in A} r[Ann\langle x\rangle]$

$r\left[\bigcup_{x \in A} Ann\langle x\rangle\right] \subseteq \bigcup_{x \in A} r[Ann\langle x\rangle]$ —(2)

From (1) and (2) $\bigcup_{x \in A} r[Ann\langle x\rangle] = r\left[\bigcup_{x \in A} Ann\langle x\rangle\right]$

Hence proved.

If I, J are ideals in a ring 'R' su that $r(I), r(J)$ are co-prime then I, J are co-prime

Proof:- Given that I, J are ideals in a ring 'R' $\ni$ $r(I), r(J)$ a co-prime.

Claim:- I, J are co-prime.

ie, $I + J = R$

Always we have $I + J \subseteq R$

Now it is enough to prove that $R \subseteq I + J$

Since $r(I), r(J)$ are co-prime, By definition, $r(I) + r(J) = R$

$\Rightarrow a + b = 1$ for some $a \in r(I)$, $b \in r(J)$ & $1 \in R$

Now $a \in r(I) \Rightarrow a^n \in I$ for some $n > 0$

Now $b \in r(J) \Rightarrow b^m \in J$ for some $m > 0$

consider
$$
\begin{aligned}
1 &= 1^{m+n} \\
&= (a+b)^{m+n} \\
&= \sum_{r=0}^{m+n} {}^{(m+n)}C_r \, a^{(m+n)-r} \, b^r \\
&\in I + J
\end{aligned}
$$

$\Rightarrow 1 \in I + J$

$\Rightarrow R \subseteq I + J$

$\therefore I + J = R$

$\therefore I, J$ are co-primes.

Hence proved.

Extension and Contraction of Ideals:- Let $f: A \rightarrow B$ be a ring homomorphism and 'I' be an ideal in a ring 'A' then $f(I)$ need not be an ideal in a ring 'B'. Then the ideal in 'B' is $B \cdot f(I)$, generated by $f(I)$ in a ring 'B' is called an "extended ideal in a ring 'B'."

ie, $B \cdot f(I) = \{ b \cdot f(x) \mid b \in B, x \in I \}$, it is denoted by I^e.

1, The extended ideal is an ideal which is generated by $f(I)$

2, If 'J' is an ideal in a ring 'B' then $f^{-1}(J) = J^c$ is always an ideal in a ring ~~'B'~~ 'A' which is called "contracted"

3, An ideal 'J' is called "extended ideal" if $J = I^e$, for some ideal 'I' in a ring 'A'.

4, An ideal 'I' is called "contracted ideal" if $I = J^c$ for some ideal J in a ring 'B'.

$$f: \overset{C}{A} \to \overset{E}{B} \qquad\qquad f: \overset{C}{A} \to \overset{E}{B}$$

$$I \longrightarrow I^e \qquad\qquad J^c \longleftarrow J$$

$$I^{ec} = (I^e)^c \leftarrow I^e \qquad\qquad J^c \to (J^c)^e = J^{ce}$$

$$I^{ec} \longrightarrow (I^{ec})^e = I^{ece} \qquad J^{cec} = (J^{ce})^c \leftarrow J^{ce}$$

Proposition:- 1.17 Let $f: A \longrightarrow B$ be a ring homomorphism and I, J are ideals in a ring 'A' and 'B'. Such that

i, $I \subseteq I^{ec}$

ii, $J^{ce} \subseteq J$

iii, $I^e = I^{ece}$

iv, $J^c = J^{cec}$

v, There is a one to one correspondence between the set of all contracted ideals and the set of all extended ideals.

Proof:- Given that $f: A \to B$ be a ring homomorphism and 'I' is an ideal in a ring 'A' and 'J' is an ideal in a ring 'B'.

i, Let us take an element $x \in I$

$\Rightarrow f(x) \in f(I) \subseteq J$

$\Rightarrow f(x) \in J$

$\Rightarrow x \in f^{-1}(J)$

$\Rightarrow x \in J^c$

$\Rightarrow x \in (I^e)^c$ (By points, 3)

$\Rightarrow x \in I^{ec}$

$\therefore I \subseteq I^{ec}$

ii, Let us take an element $x \in J^{ce}$

$\Rightarrow x \in (J^c)^e$

$\Rightarrow x \in I^e$ $(\because J^c = I)$

$\Rightarrow x \in J$ $(\because I^e = J)$

$\therefore J^{ce} \subseteq J$

iii, From i, we have $I \subseteq I^{ec}$

$\Rightarrow I^e \subseteq (I^{ec})^e$

$\Rightarrow I^{e} \subseteq I^{ece}$ — (1)

from ii, we have $J^{ce} \subseteq J$ where $J = I^{e}$

$$(I^{e})^{ce} \subseteq I^{e}$$

$$I^{ece} \subseteq I^{e} \text{ — (2)}$$

From (1) & (2) $\therefore I^{e} = I^{ece}$

iv, From ii, we have $J^{ce} \subseteq J$

$$(J^{ce})^{c} \subseteq J^{c} \text{ — (a)}$$

$$J^{cec} \subseteq J^{c} \text{ — (3)}$$

From (i) we have $I \subseteq I^{ec}$ $(\because I = J^{c})$

$$J^{c} \subseteq (J^{c})^{ec}$$

$$\Rightarrow J^{c} \subseteq J^{cec} \text{ — (4)}$$

From (3) and (4), we have $J^{c} = J^{cec}$

From (i) ii, iii & iv, we conclude that there is a one to one correspondence between the set of all contracted ideals in ring 'A' into the set of all extended ideals in a ring 'B'

Problems:-

1, Let 'R' be the ring every nilpotent element is a zero divisor

Sol:- Let 'R' be the ring.

Take $x \in R$ be the nilpotent element.

By def; $\exists\, n \in Z^{+} \ni x^{n} = 0$

Suppose $n = 1$ then $x = 0$ and hence it is a zero divisor.

Suppose $n > 1$ then $x^{n} = 0 \Rightarrow x \cdot x^{n-1} = 0$

Hence 'x' is a zero-divisor.

2, Let 'x' be a nilpotent element in a ring 'R' then show that $1-x$ is a unit in a ring 'R'. Hence deduce that sum of the nilpotent element and a unit element is also a unit element

Sol:- i, Let us take 'R' be the ring and $x \in R$ is a nilpotent element.

By def $\exists\, n \in Z^{+} \ni x^{n} = 0$

$(1^2 - x^2) = (1-x)(1+x)$
$(1^3 - x^3) = (1-x)(1+x+x^2)$
$(1-x^n) = (1-x)(1+x+x^2+\dots+x^{n-1})$

Now consider, $1 = 1 - 0$

$$= 1 - x^{n}$$

$$= (1-x)\cancel{x^{n-1}}(1+x+x^2+x^3+\dots+x^{n-1}) \text{ (By def, unit)}$$

$\Rightarrow 1-x$ is a unit.

ii) Let us take 'x' be the nilpotent element and 'a' be the unit element.

By def., $\exists\, n \in z^{+} \ni x^{n} = 0$

By def., $\exists\, b \neq 0 \in R \ni ab = 1$

Now consider $(x+a)\cdot b = xb + ab$

$= xb + 1$

$= 1 + xb$

$= 1 - (-xb)$

Since 'x' is a nilpotent element $\Rightarrow xb$ is also nilpotent element.

$\therefore$ we have $(x+a)\, b = 1 - (-xb)$

from i, R.H.S. $1-(-xb)$ is unit. ($\because x = -xb$)

$\Rightarrow (x+a)b$ is a unit

$\Rightarrow x+a$ is a unit.

3, Let 'R' be the ring and $x \in R$ then $\langle x \rangle = R \Leftrightarrow x$ is a unit in a ring 'R'.

Sol:- Let us take 'R' be the ring and $x \in R$

i, now take $\langle x \rangle = R$

$\Rightarrow 1 \in \langle x \rangle$

$\Rightarrow 1 \in xR$

$\Rightarrow 1 = xy$ for some $y \in R$

$\therefore x$ is a unit in 'R'.

conversely, suppose that 'x' is a unit.

By def., $\exists\, y \neq 0 \in R \ni xy = 1$

ie, $1 = xy \in xR$

$\Rightarrow 1 \in xR$

$\Rightarrow 1 \in \langle x \rangle$

$\Rightarrow R = \langle x \rangle$

4, In the ring $A[x]$ Jacobison radical is equal to nil~~poten~~ radical.

Sol:- Take the polynomial ring $A[x]$

Also take 'N' be the nilradical of the ring $A[x]$ and 'J' be the Jacobson radical of the ring $A[x]$

we know that $N \subseteq J$

Now it is enough to prove that $J \subset N$

Now let us take an element $f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n \in$

Recall:- Theorem - 1.9

By the above recall $1+f(x)\cdot g(x)$ is a unit in a ring $g(x) \in A[x]$

$\Rightarrow 1+f(x)\cdot x$ is a unit where $g(x)=x$

$\Rightarrow 1+(a_0+a_1x+a_2x^2 \dots a_nx^n)x$ is a unit

$\Rightarrow 1+(a_0x+a_1x^2+a_2x^3+\dots+a_nx^{n+1})$ is a unit.

$\Rightarrow 1-[-(a_0x+a_1x^2+a_2x^3+\dots+a_nx^{n+1})]$ is a unit

$\because a_0, a_1, a_2 \dots a_n$ are nilpotent elements.

$\Rightarrow f(x)$ is a nilpotent element (or) $a_0x+a_1x^2+a_2x^3+\dots+a_nx^{n+1}$ nilpotent element

$\Rightarrow f(x) \in N$

$\therefore J \subseteq N$

$\therefore J = N$

$\therefore$ Jacobson radical is equal to nilradical.

5, Show that $[N:P]$ is an ideal in a ring 'A' where N,P ideals in a ring 'A'.

Sol:- Let us take $x, y \in [N:P]$

$\Rightarrow x \in [N:P]$ By definition, $xP \subseteq N$

$\Rightarrow y \in [N:P]$ By definition, $yP \subseteq N$

$\Rightarrow xP+yP \subseteq N$

$\Rightarrow (x+y)+P \subseteq N$

$\Rightarrow (x+y) \in [N:P]$

Now take $x \in [N:P]$ and $a \in A$

$\Rightarrow xP \subseteq N$

$\Rightarrow xy \in N$ for some $y \in P$

Now $xy \in N$ and $a \in A$ and N is an ideal $\Rightarrow a(xy) \in N$

$\Rightarrow (ax)y \in N$

$\Rightarrow (ax)P \subseteq N$

$\therefore [N:P]$ is an ideal in a ring 'A'. $\Rightarrow ax \in [N:P]$